Stat 821 Homework 7 Solution

Question 1 (Problem 2.7)

(a) $X_i \stackrel{iid}{\sim} U(0,\theta)$, $\delta_n = \frac{(n+1)X_n}{n}$ is UMVUE of θ, MLE is X_n.

$$MSE(\delta_n) = E(\delta-\theta)^2 = Var\delta_n = (\frac{n+1}{n})^2 Var(X_{(n)})$$

$$F_{X_{(n)}}(x) = [P(X_i < x)]^n = (\frac{x}{\theta})^2$$

$$f_{X_{(n)}}(x) = \frac{n}{\theta}(x/\theta)^{n-1} = \frac{n}{\theta^n}x^{n-1}, \quad 0 \le x < \theta$$

Let $Y_i = X_i/\theta$, then

$$Y_i \sim U(0,1) \Rightarrow Y_{(n)} \sim Beta(n,1)$$

. thus

$$Var[X_{(n)}] = Var(\theta Y_{(n)}) = \frac{n\theta^2}{(n+1)^2(n+2)}$$

$$\Rightarrow \quad MSE(\delta_n) = \frac{\theta^2}{n(n+2)}$$

$$\begin{aligned} MSE(X_{(n)}) &= E(X_{(n)}-\theta)^2 \\ &= E(X_{(n)} - E(X_{(n)}) + E(X_{(n)}) - \theta)^2 \\ &= Var(X_{(n)}) + (\frac{\theta}{n+1})^2 \\ &= \frac{2\theta^2}{(n+1)(n+2)} \end{aligned}$$

(b)

$$\lim_{n\to\infty} \frac{E(X_{(n)}-\theta)^2}{E(\delta_n-\theta)^2} = \lim_{n\to\infty} \frac{\frac{2\theta^2}{(n+1)(n+2)}}{\frac{\theta^2}{n(n+2)}} = 2$$

Question 2 (Problem 3.5)

(a)

$$f(x) = p^{1-x}q^x$$

If $x = 0$, $l(p) = p$, which is maximized when $p = 2/3$; If $x = 1$, $l(p) = 1 - p$, which is maximized when $p = 1/3$. Thus

$$\bar{p}_{mle} = \begin{cases} 2/3 & x = 0 \\ 1/3 & x = 1 \end{cases}$$

(b)

$$E(\bar{p} - p)^2 = \frac{1}{9}(3p^2 - 3p + 1)$$

$$E(\delta(X) - p)^2 = E(1/2 - p)^2 = \frac{1}{4}(1 - 2p)^2$$

$$E(\bar{p}-p)^2 - E(\delta(X)-p)^2 = \frac{1}{9}(3p^2-3p+1) - \frac{1}{4}(1-2p)^2 = -\frac{2}{3}[(p-1/2)^2 - 1/24]$$

$$p \in [1/3, 2/3] \quad \Rightarrow \quad (p - 1/2)^2 - 24 \in [-1/24, -1/72]$$

Thus

$$E(\bar{p} - p)^2 > E(\delta(X) - p)^2, \quad \forall \frac{1}{3} \leq p \leq \frac{2}{3}$$

i.e. the MSE of MLE is uniformly larger than that of $\delta(X) = 1/2$.

Question 3 (Problem 6.14)

(a) Let $y_i = |x_i|$, then

$$f(y_i) = \frac{2}{\sqrt{2\pi\sigma^2}} e^{-\frac{y_i^2}{2\sigma^2}} \qquad y_i > 0$$

$$E(Y_i) = \int_0^\infty f(y_i) y_i dy_i = \frac{2\sigma}{\sqrt{2\pi}}[-e^{-\frac{y_i^2}{2\sigma^2}}]|_0^\infty = \sqrt{\frac{2}{\pi}}\sigma$$

$$\Rightarrow \frac{\sum y_i}{n} \xrightarrow{P} \sqrt{\frac{2}{\pi}}\sigma$$

$$\Rightarrow \sqrt{\frac{\pi}{2}} \frac{\sum |x_i|}{n} \xrightarrow{P} \sigma$$

Thus

$$k = \sqrt{\frac{\pi}{2}} \quad \Leftrightarrow \quad \delta_n \text{ is constant of } \sigma$$

(b) From (a),

$$\sqrt{n}(\bar{y} - \sqrt{\frac{2}{\pi}}\sigma) \to N(0, Vary_i)$$

$$EY_i^2 = EX_i^2 = \sigma^2 \qquad VarY_i = (1 - 2/\pi)\sigma^2$$

$$\sqrt{n}(\delta_n - \sigma) \to N(0, \frac{\pi}{2}Vary_i) \Rightarrow \tau_1^2 = (\pi/2 - 1)\sigma^2$$

$$Z_i = X_i^2 \quad EZ_i = \sigma^2 \quad VarZ_i = 2\sigma^4$$

since $\frac{X^2}{\sigma^2} \sim \chi_1^2$. Thus

$$\sqrt{n}(\bar{Z} - \sigma^2) \to N(0, 2\sigma^4)$$

By delta method,

$$\sqrt{n}(\sqrt{\bar{Z}} - \sigma) \to N(0, \frac{2\sigma^4}{4\sigma^2}) \ \Rightarrow \ \tau_2^2 = \frac{\sigma^2}{2}$$

$$\text{ARE} \quad e_{2,1} = \frac{(\pi/2 - 1)\sigma^2}{\sigma^2/2} = \pi - 2$$